MA1 – zkouška 14.1.2021 (dopoledne)

① $I = \int \frac{x}{1+x^4} + \frac{4-\sqrt{x}}{x(x+2\sqrt{x}+2)}\,dx = I_1 + I_2$

1b

interval : $[0,+\infty)$

$I_1 = \int \frac{x}{1+x^4}\,dx = \left| \begin{matrix} x^2 = t \\ 2x\,dx = dt \\ \text{subst. } \cancel{1b}\ b \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{1+t^2} =$ 0,5b

$\frac{1}{2}\operatorname{arctg} t + C = \frac{1}{2}\operatorname{arctg}(x^2) + C$

~~...~~

$I_2 = \int \frac{4-\sqrt{x}}{x(x+2\sqrt{x}+2)}\,dx = \left| \begin{matrix} \sqrt{x} = t \\ x = t^2 \\ dx = 2t\,dt \end{matrix} \right| = \int \frac{(4-t)\cdot 2\cancel{t}}{t^{\cancel{2}}(t^2+2t+2)}\,dt$

subst. 2b

$\overset{\vee}{=} 2\int \frac{4-t}{t(t^2+2t+2)}\,dt = 2\left(2\int \frac{1}{t}dt - \int \frac{2t+5}{t^2+2t+2}\,dt\right)$

$= 4\ln|t| - 2\left(\int \frac{2t+2}{t^2+2t+2} + 3\int \frac{1}{(t+1)^2+1}\,dt\right) =$

$= 4\ln|t| - 2\ln(t^2+2t+2) - 6\operatorname{arctg}(t+1) + C =$

$= 4\ln\sqrt{x} - 2\ln(x+2\sqrt{x}+2) - 6\operatorname{arctg}(\sqrt{x}+1) + C$

integrace 2b (0,5 + 4·0,5)

Rozklad : $\frac{2(4-t)}{(t^2+2t+2)t} = \frac{A}{t} + \frac{Bt+C}{t^2+2t+2}$

$8 - 2t = A(t^2+2t+2) + Bt^2 + Ct$

$$\begin{aligned} &1. \quad A + B = 0 &&\Rightarrow B = -4 \\ &2A + C = -2 &&\Rightarrow C = -10 \\ &2A = 8 &&\Rightarrow \underline{A = 4} \end{aligned}$$

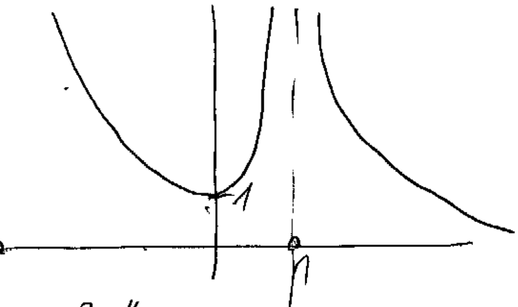
$\overset{\vee}{\underline{B=-2}}$ rozklad 2b

$\underline{C=-5}$

$\underline{A=2}$

(2) $f(x) = \dfrac{e^{-2x}}{(x-1)^2}$

odhad grafu:



a) $D_f = R \setminus \{1\} = (-\infty, 1) \cup (1, +\infty)$, $f(0) = 1$ } 1b
$f(x) > 0$ v D_f, spojitá v D_f

1b { $\lim\limits_{x\to+\infty} \dfrac{e^{-2x}}{(x-1)^2} = „\dfrac{0}{\infty}" = 0$, $\lim\limits_{x\to 1} \dfrac{e^{-2x}}{(x-1)^2} = „\dfrac{e^{-2}}{0+}" = +\infty$ 1b.

$\lim\limits_{x\to-\infty} \dfrac{e^{-2x}}{(x-1)^2} = „\dfrac{\infty}{\infty}" \underset{2\times l'H}{=} +\infty$ (velr „útvalo")

b) $f'(x) = \dfrac{-2e^{-2x}(x-1)^2 - 2e^{-2x}\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}} = \dfrac{-2e^{-2x}(x-1+1)}{(x-1)^3} = \dfrac{-2e^{-2x} \cdot x}{(x-1)^3}$ f' 1b

$f'(x) = 0 \Leftrightarrow x = 0$ $\Rightarrow$ v $x=0$ je ostré lok. minimum 0,5b

f': $-$ | 0 | $+$ | 1 | $-$
$\searrow$ 0 $\nearrow$ 1 $\searrow$
vyšetření 1b

glob. min fce nemá ($f(x) > 0$ a $\lim\limits_{x\to+\infty} f = 0$) } 0,5b
glob. max nemá ($\lim\limits_{x\to 1} f = +\infty$)

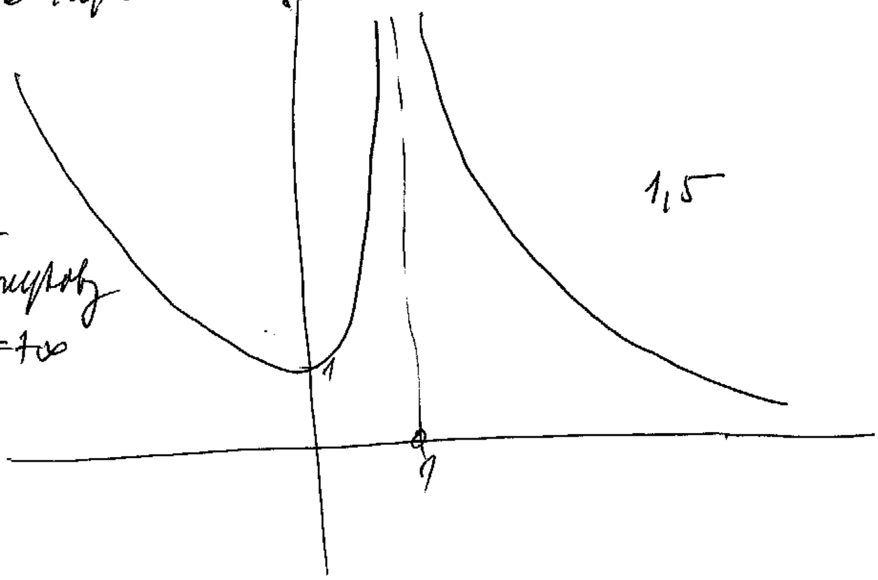
c) $f''(x) = -2\, \dfrac{(e^{-2x} - 2xe^{-2x})(x-1)^{\cancel{3}} - e^{-2x} \cdot x\, 3\cancel{(x-1)^2}}{(x-1)^{\cancel{6}\,4}} = \dfrac{+2e^{-2x}}{(x-1)^4}\left[(2x-1)(x-1) + 3x\right]$

f'' 1,5b $= \dfrac{2e^{-2x}}{(x-1)^4}(2x^2+1)$, $f''(x) > 0$ v D_f $\Rightarrow$

$\Rightarrow$ $f(x)$ je konvexní v $(-\infty, 1)$ i v $(1, +\infty)$ 1,5b
f nemá inflexní body

graf:

asymptoty: $y = 0$ v $+\infty$
$x = 1$



1,5

(?) šikmá v $-\infty$: $y = ax + b$ 0,5 asymptoty

$a = \lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to-\infty} \dfrac{e^{-2x}}{x(x-1)^2} = +\infty$

($3\times$ l'H.)

$\Rightarrow$ ~~f nemá šikmou~~
asymptotu

(3) $S(\omega)$, ω ohran. $y = \ln x$ a $y = \ln^2 x$

1) $\ln x = \ln^2 x \iff \ln x = 0 \vee \ln x = 1$ $\quad$ j. $x \in \langle 1, e\rangle$
$x = 1$ a $x = e$ $\Big\} \Rightarrow$

$0 < \ln x < 1$ v $(1, e) \Rightarrow \ln^2 x < \ln x$ v $(1, e)$

$S(\omega) = \int_1^e (\ln x - \ln^2 x)\,dx = \int_1^e \ln x - \int_1^e \ln^2 x\,dx$

(ex. jako R i N) $\rightarrow$ nerdel 3b — $\langle 1, e\rangle$ 1b
$\ln x \geq \ln^2 x$ 1,5b / zohádku
S -0,5b

Výpočet integrálů:

$I_1 = \int_1^e \ln x\,dx \underset{M}{=} \left| \begin{matrix} u' = 1 & u = x \\ v = \ln x, & v' = 1 \end{matrix} \right| = [x \ln x]_1^e - \int_1^e dx =$

1,0b $\quad$ 1b

$[x\ln x - x]_1^e = e - (e-1) = 1$ ~~drsnami 0,5b~~

$I_2 = \int_1^e \ln^2 x\,dx \underset{M}{=} \left| \begin{matrix} u' = 1 & u = x \\ v = \ln^2 x, & v' = 2\ln x \cdot \frac{1}{x} \end{matrix} \right| =$

2b

$= [x \ln^2 x]_1^e - 2\int_1^e \ln x\,dx = [x\ln^2 x]_1^e - 2\underbrace{[x\ln x - x]_1^e}_{=1}$

$= e - 2$

j. $S(\omega) = I_1 - I_2 = 1 - (e-2) = 3 - e$

(4) $y' = \frac{x}{\sqrt{x^2-1}}(2-y)$ a) $x\in(-\infty,-1)$, $x\in(1,+\infty)$

(i) $y(x)=2$, $x\in$ —"— stac. řešení 1b

(ii) $y(x)\neq 2$ separace $\int \frac{dy}{2-y} = \int \frac{x}{\sqrt{x^2-1}}\,dx$ 1b (bez $y\neq 2$ 0,5b)

ř. $-\ln|2-y| = \sqrt{x^2-1} + C$, $C\in\mathbb{R}$ integrace 1+1

$\ln|y-2| = -\sqrt{x^2-1} + C$, $C\in\mathbb{R}$ úprava (y(x)) 3b

$|y-2| = e^{-\sqrt{x^2-1}}\cdot e^{C}$, $C\in\mathbb{R}$ nejasná K ~~1b~~ -0,5b ?

bez intervalů -0,5b

$y = 2 + K e^{-\sqrt{x^2-1}}$, $K\neq 0$, $x\in(-\infty,-1) \vee x\in(1,+\infty)$, „špatné" odstranění -1,5

(i) a (ii) : $y_{OR} = 2 + K e^{-\sqrt{x^2-1}}$, $K\in\mathbb{R}$, —"—

b) počáteční úlohy:

(i) $y(-2)=0$: 0,5b

$0 = 2 + K e^{-\sqrt{3}} \Rightarrow K = -2e^{\sqrt{3}}$

ř. $y(x) = 2 - 2e^{\sqrt{3}-\sqrt{x^2-1}}$, $x\in(-\infty,-1)$

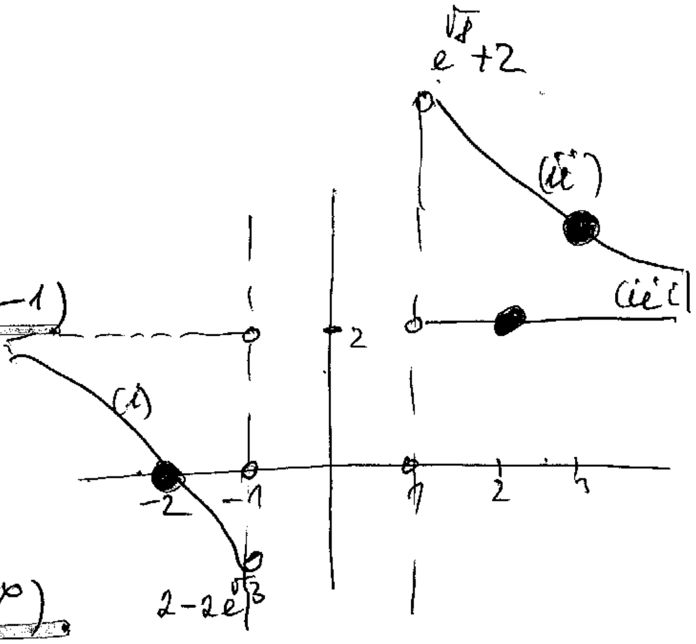
(ii) $y(3)=3$:

$3 = 2 + K e^{-\sqrt{8}} \Rightarrow K = e^{\sqrt{8}}$

$y(x) = 2 + e^{\sqrt{8}-\sqrt{x^2-1}}$, $x\in(1,+\infty)$ 0,5

(iii) $y(2)=2$

stac. řešení $y(x)=2$, $x\in(1,+\infty)$ 0,5b



3 × 0,5

Úlohy teoretické

1

b) $A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & -2 & 3 & 0 \\ -1 & 2 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & -4 & 5 & -1 \\ 0 & -4 & 5 & -1 \\ 0 & 4 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim$

$\sim \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$ matice ekvivalentní s A má hodnost 3 $\Rightarrow$

$\Rightarrow h(A) = 3$

c) Řešení soustavy $A \cdot x = 0$

$$x_1 + x_2 - x_3 + x_4 = 0$$
$$2x_2 - x_3 - x_4 = 0$$
$$x_3 - x_4 = 0$$

$x_4 = t$, $x_3 = t$

$x_2 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2} \cdot 2t = t$

$x_1 = -x_2 + x_3 - x_4 = -t$

ř. $(x_1, x_2, x_3, x_4) = t(-1, 1, 1, 1)$, $t \in R$

2

(i) $\lim\limits_{x \to 0} \dfrac{e^{3x^2} - \cos x}{x^2} \underset{l'H}{=} \text{"}\dfrac{0}{0}\text{"} = \lim\limits_{x \to 0} \dfrac{e^{3x^2} \cdot 6x + \sin x}{2x} = \lim\limits_{x \to 0} \left(3 e^{3x^2} - \dfrac{1}{2} \dfrac{\sin x}{x}\right) =$

$= 3 + \frac{1}{2} = \frac{7}{2}$, ř. f lze spojitě dodef. v $a = 0$: $f(0) = \frac{7}{2}$

(ii) $\lim\limits_{x \to 0\pm} \dfrac{\operatorname{arctg} x}{x^2} = \text{"}\dfrac{0}{0}\text{"} = \lim\limits_{x \to 0\pm} \underbrace{\dfrac{\operatorname{arctg} x}{x}}_{\to 1} \cdot \dfrac{1}{x} = \pm\infty \Rightarrow$

$\Rightarrow$ g(x) nelze spojitě dodef. v bodě $a = 0$

③ $f(x) = \cos(e^{4x} - 1), \quad a = 0$

$f(0) = \cos 0 = 1$

$f'(x) = -\sin(e^{4x} - 1) \cdot 4e^{4x} \Rightarrow f'(0) = 0$

$f''(x) = -\cos(e^{4x} - 1) \cdot (4e^{4x})^2 - \sin(e^{4x} - 1) \cdot 4 \cdot e^{4x} \cdot 4 \Rightarrow$

$\Rightarrow f''(0) = -16$

1. $T_2(x) = 1 - \frac{16}{2}(x^2) = 1 - 8x^2$